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On the Continuity of the Kernel of Invex Functions

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For an invex function, it usually suffices that a kernel merely exists, with no need for any restrictions on the kernel. However, this is not always the case. We present here several situations where some degree of continuity is required and establish conditions sufficient for both existence and non-existence of a continuous kernel.

1. INTRODUCTION

Invex functions, which were first studied by Hanson [3] and named by Craven [2], have the useful property that they provide a generalisation of convexity allowing the Kuhn–Tucker conditions for a nonlinear program to be both necessary and sufficient. Most subsequent papers on these functions have been more or less concerned with establishing this property for a specialized program or an abstract space or with some relaxation of smoothness. One common feature in this literature is that invexity with respect to some kernel (usually called η) is assumed, then manipulations are carried out with no regard, and indeed no need of regard, for the analytic properties of this kernel. However, there are occasions when assumptions about the kernel itself need to be made. Here, in Section 2 we describe two situations when continuity must be imposed; in Section 3, we establish conditions guaranteeing the existence of a continuous kernel; Section 4 presents a sufficient condition for non-existence in the one-dimensional case and gives examples.

The usual definition of invexity for a differentiable function in Euclidean space is as follows:

DEFINITION. Let $C \subset \mathbb{R}^n$ be an open set and let $f: C \to \mathbb{R}$ be differentiable with gradient denoted ∇f (we will use f' when $n = 1$). Then f is

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said to be *invex* if $\exists$ a function η: $C \times C \to \mathbb{R}^n$ (called the *kernel*) such that $f(x) - f(u) \geq \eta(x, u)^T \nabla f(u)$ for all $x, u \in C$. We say that f is invex with respect to η.

2. EXAMPLES REQUIRING A CONTINUOUS KERNEL

EXAMPLE 2.1. In Parida *et al*. [4], a variational-like inequality problem is examined, and one of the applications of their main theorem is on an invex mathematical program.

The variational-like inequality problem treated is:

> Given a closed convex set $K \subset \mathbb{R}^n$, and two continuous maps F: $K \to \mathbb{R}^n$ and η: $K \times K \to \mathbb{R}^n$, find $\bar{x} \in K$ such that $F(\bar{x})^T \eta(x, \bar{x}) \geq 0$ $\forall x \in K$.

For the application to mathematical programming, they assume f is a continuously differentiable real-valued function on K (or rather some open set containing K), invex with respect to η, and take $F = \nabla f$. Consider the program:

$$\text{Min } f(x) \text{ subject to } x \in K. \tag{PSK}$$

It is shown that if $\bar{x}$ solves the variational-like inequality problem, then $\bar{x}$ is an optimal solution of the program (PSK). The existence of a solution to the variational-like inequality problem depends on the continuity of η allowing the Kakutani fixed-point theorem to be invoked.

Yang and Chen [6] studied the pre-variational inequality problem (another term for variational-like inequality) and proved the existence of solutions under several alternative conditions on η; namely, pre-coercivity, normality, and regularity. Normality requires differentiability, and regularity will hold given continuity, so the results we present here will also be applicable in these authors' theorems.

EXAMPLE 2.2. Ponstein [5] established six equivalent definitions of quasi-convexity, of which two apply to differentiable functions. It is of interest whether the equivalence for these two can be extended to quasi-invexity. We will prove here that the equivalence is dependent on a continuity property for the kernel.

First, we repeat Ponstein's result: Assume that f: $C \to \mathbb{R}$ is differentiable, where C is a convex subset of $\mathbb{R}^n$. f is quasi-convex if either

$$f(x_2) \leq f(x_1) \Rightarrow (x_2 - x_1)^T \nabla f(x_1) \leq 0, \tag{A}$$

or equivalently,

$$f(x_2) < f(x_1) \Rightarrow (x_2 - x_1)^T \nabla f(x_1) \leq 0. \tag{B}$$

Quasi-invex functions form a wider class: f is said to be quasi-invex if there exists a function $\eta\colon C \times C \to \mathbb{R}^n$ such that

$$f(x_2) \le f(x_1) \Rightarrow \eta(x_2, x_1)^T \nabla f(x_1) \le 0. \tag{A1}$$

Below, we give a condition on η to guarantee that (A1) is equivalent to

$$f(x_2) < f(x_1) \Rightarrow \eta(x_2, x_1)^T \nabla f(x_1) \le 0. \tag{B1}$$

This result also subsumes that of Ponstein if we take $\eta(x_2, x_1) = x_2 - x_1$.

THEOREM 2.1. *If the function η satisfies $\eta(x_2, \cdot)$ continuous at x_1 whenever $f(x_2) = f(x_1)$ and f is continuously differentiable, then conditions* (A1) *and* (B1) *are equivalent.*

Proof. Clearly, if (A1) holds then (B1) holds.

Conversely, if (B1) holds we need only establish that $f(x_2) = f(x_1) \Rightarrow \eta(x_2, x_1)\nabla f(x_1) \le 0$. Assume that $\exists x_1, x_2 \in C$ (not necessarily distinct) such that $f(x_2) = f(x_1)$ and $\eta(x_2, x_1)^T \nabla f(x_1) > 0$. Then, by continuity of f, $\exists \bar{\lambda} > 0$, such that $\forall \lambda < \bar{\lambda}$, $\lambda \ne 0$, we have $f(x_1 + \lambda\eta(x_2, x_1)) > f(x_1) = f(x_2)$.

By (B1), this gives $\eta(x_2, x_1 + \lambda\eta(x_2, x_1))^T \nabla f(x_1 + \lambda\eta(x_2, x_1)) \le 0$. Taking limits as $\lambda \downarrow 0$, we obtain by continuity of $\eta(x_2, \cdot)$ and ∇f that $\eta(x_2, x_1)^T \nabla f(x_1) \le 0$, a contradiction. Thus, if (B1) holds then (A1) holds. ∎

3. SUFFICIENT CONDITIONS FOR A CONTINUOUS KERNEL

Ben-Israel and Mond [1] have shown that $f\colon C \to \mathbb{R}$ is invex with respect to $\eta\colon C \times C \to \mathbb{R}^n$ iff $\forall x \in X$ and $\forall u \in C$ such that $\nabla f(u) \ne 0$,

$$\eta(x, u) \in \left\{ \frac{(f(x) - f(u))\nabla f(u)}{\nabla f(u)^T \nabla f(u)} + v;\ v^T \nabla f(u) \le 0,\ v = v(x, u) \right\}.$$

The question to be dealt with here is: Under what conditions on f can a continuous η be chosen subject to the above constraint?

To demonstrate that the answer is not immediately obvious, consider the easiest choice of η; namely

$$\eta(x, u) = \begin{cases} \dfrac{(f(x) - f(u))\nabla f(u)}{\nabla f(u)^T \nabla f(u)}, & \nabla f(u) \ne 0, \\ w, & \nabla f(u) = 0, \end{cases}$$

for some choice of $w \in \mathbb{R}^n$.

In the case of $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$, this choice gives $\eta(x,u) = (x^2 - u^2)/2u$ for $u \neq 0$. For a fixed $x \in \mathbb{R}$, $x \neq 0$, $\lim_{u \to 0} \eta(x,u)$ does not exist, so no choice of w makes $\eta(x, \cdot)$ continuous at 0.

An alternative choice of η is

$$\eta(x,u) = \begin{cases} 0, & f(x) \geq f(u) \\ \dfrac{(f(x) - f(u))\nabla f(u)}{\nabla f(u)^T \nabla f(u)}, & f(x) < f(u), \end{cases}$$

which is formed by choosing v so that $v^T \nabla f(u) = f(u) - f(x)$ whenever $f(u) \leq f(x)$ with $\nabla f(u) \neq 0$, choosing $v = 0$ whenever $f(u) > f(x)$, and setting $\eta(x,u) = 0$ when $\nabla f(u) = 0$.

In our simple example, this gives

$$\eta(x,u) = \begin{cases} \dfrac{x^2 - u^2}{2u}, & |u| > |x| \\ 0, & |u| \leq |x| \end{cases}$$

which is continuous on $\mathbb{R}^2$.

In fact, the second choice of η can be shown to be continuous subject to a limit condition on f.

THEOREM 3.1. *Let $f: C \to \mathbb{R}$ be continuously differentiable and invex. The function $\eta: C \times C \to \mathbb{R}^n$, with respect to which f is invex, defined by*

$$\eta(x,u) = \begin{cases} 0, & f(x) \geq f(u) \\ \dfrac{(f(x) - f(u))\nabla f(u)}{\nabla f(u)^T \nabla f(u)}, & f(x) < f(u), \end{cases}$$

is continuous if, given u such that $\nabla f(u) = 0$, then for any sequence $\{u^n\}$, $u^n \to u$, $\nabla f(u^n) \neq 0$, we have

$$\lim_{n \to \infty} \frac{|f(u) - f(u^n)|}{\|\nabla f(u^n)\|} = 0,$$

where $\|\cdot\|$ is the Euclidean norm.

Proof. Let $(x,u) \in C \times C$ and assume $\{x^n\}$ and $\{u^n\}$ are sequences such that $(x^n, u^n) \in C \times C$, $x^n \to x$, and $u^n \to u$. We want to show that $\lim_{n \to \infty} \eta(x^n, u^n) = \eta(x,u)$. Three separate cases must be considered: (a) $f(x) < f(u)$, (b) $f(x) > f(u)$, and (c) $f(x) = f(u)$.

(a) From the definition of η, we have

$$\eta(x,u) = \frac{(f(x) - f(u))\nabla f(u)}{\nabla f(u)^T \nabla f(u)}.$$

By continuity of f, $\exists N \in \mathbb{N}$ such that, $\forall n \geq N$, $f(x^n) < f(u^n)$. Therefore, for $n \geq N$,

$$\eta(x^n, u^n) = \frac{(f(x^n) - f(u^n))\nabla f(u^n)}{\nabla f(u^n)^T \nabla f(u^n)}.$$

By continuity of ∇f, $\lim_{n\to\infty} \eta(x^n, u^n) = \eta(x, u)$.

(b) By hypothesis, $\eta(x, u) = 0$. Again, by continuity of f, $\exists$ an $N \in \mathbb{N}$ such that $\forall n \geq N$, $f(x^n) > f(u^n)$, and thus $\eta(x^n, u^n) = 0$. Therefore, $\lim_{n\to\infty} \eta(x^n, u^n) = \eta(x, u)$.

(c) By hypothesis, $\eta(x, u) = 0$. There are two sub-cases to treat: (c1) $\nabla f(u) \neq 0$ and (c2) $\nabla f(u) = 0$.

(c1) By continuity of f and ∇f, $\forall \epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $\forall n \geq N$,

$$|f(x^n) - f(x)| < \epsilon/2, |f(u^n) - f(u)| < \epsilon/2, \qquad \text{and} \qquad \nabla f(u^n) \neq 0.$$

Now, for $n \geq N$, if $f(x^n) \geq f(u^n)$, then $\eta(x^n, u^n) = 0$, and if $f(x^n) < f(u^n)$, then

$$\eta(x^n, u^n) = \frac{(f(x^n) - f(u^n))\nabla f(u^n)}{\nabla f(u^n)^T \nabla f(u^n)}.$$

We also have $|f(x^n) - f(u^n)| < \epsilon$. Hence, for $f(x^n) < f(u^n)$,

$$\begin{aligned}
\|\eta(x^n, u^n)\| &= \left\| \frac{(f(x^n) - f(u^n))\nabla f(u^n)}{\nabla f(u^n)^T \nabla f(u^n)} \right\| \\
&= \frac{\|(f(x^n) - f(u^n))\nabla f(u^n)\|}{\|\nabla f(u^n)\|^2} \\
&= \frac{|f(x^n) - f(u^n)|\,\|\nabla f(u^n)\|}{\|\nabla f(u^n)\|^2} \\
&< \frac{\epsilon}{\|\nabla f(u^n)\|}.
\end{aligned}$$

As this holds $\forall \epsilon > 0$ and ∇f continuous, then $\lim_{n\to\infty} \|\eta(x^n, u^n)\| = 0$, so that $\lim_{n\to\infty} \eta(x^n, u^n) = 0 = \eta(x, u)$.

(c2) If $f(x^n) \geq f(u^n)$ then $\eta(x^n, u^n) = 0$. If $f(x^n) < f(u^n)$, then

$$\eta(x^n, u^n) = \frac{(f(x^n) - f(u^n))\nabla f(u^n)}{\nabla f(u^n)^T \nabla f(u^n)}$$

and so

$$\|\eta(x^n, u^n)\| = \frac{|f(x^n) - f(u^n)|}{\|\nabla f(u^n)\|}.$$

Note that $\nabla f(u) = 0$ and $f(x) = f(u)$ imply that x and u are global minimizers, so that when $f(x^n) < f(u^n)$, we have $f(u) = f(x) \leq f(x^n) < f(u^n)$. This gives

$$|f(u) - f(u^n)| \geq |f(x^n) - f(u^n)|$$

and hence

$$\|\eta(x^n, u^n)\| \leq \frac{|f(u) - f(u^n)|}{\|\nabla f(u^n)\|}.$$

Now, if there exists an $N \in \mathbb{N}$ such that $\forall n \geq N$, $f(x^n) \geq f(u^n)$, then we immediately have $\lim_{n \to \infty} \eta(x^n, u^n) = 0 = \eta(x, u)$.

Otherwise, there exists a subsequence $\{u^{n_i}\}$ of $\{u^n\}$ such that $u^{n_i} \to u$, $f(x^{n_i}) < f(u^{n_i})$, and $\nabla f(u^{n_i}) \neq 0$.

By the limit hypothesis,

$$\lim_{n_i \to \infty} \|\eta(x^{n_i}, u^{n_i})\| \leq \lim_{n_i \to \infty} \frac{|f(u) - f(u^{n_i})|}{\|\nabla f(u^{n_i})\|} = 0.$$

Therefore, $\lim \eta(x^n, u^n) = 0 = \eta(x, u)$. ∎

There is a simple sufficient second-order condition which guarantees the limit property required in Theorem 3.1.

THEOREM 3.2. *Let f: $C \to \mathbb{R}$ be invex and assume $\nabla f(u) = 0$. If f is twice continuously differentiable in some open neighbourhood of u and $\nabla^2 f(u)$ is positive definite, then for any sequence $\{u^n\}$, $u^n \in C$, $u^n \to u$, $\nabla f(u^n) \neq 0$, we have*

$$\lim_{n \to \infty} \frac{|f(u) - f(u^n)|}{\|\nabla f(u^n)\|} = 0.$$

Proof. As f is twice differentiable in some open neighbourhood of u, and u is a local minimizer with $\nabla^2 f(u)$ positive definite, then by continuity of $\nabla^2 f$, there exists some $\epsilon > 0$ such that for all $x \in N(u, \epsilon)$ (the open ball of radius ϵ centered at u), f is twice differentiable at x and $\nabla^2 f(x)$ is positive semi-definite.

Now, consider $x \in N(u, \epsilon)$, $x \neq u$, and define $g\colon [0, 1] \to \mathbb{R}$ by $g(t) = f(u + t(x - u))$. g is twice differentiable, and its derivatives are given by

$$g'(t) = (x - u)^T \nabla f(u + t(x - u))$$
$$g''(t) = (x - u)^T \nabla^2 f(u + t(x - u))(x - u).$$

Let $t \in (0, 1]$. By the Mean Value Theorem, $\exists \xi \in (0, t]$ such that

$$g'(\xi) = \frac{g(t) - g(0)}{t},$$

that is, $g(t) - g(0) = tg'(\xi)$. But, as $\nabla^2 f$ is positive semi-definite on $N(u, \epsilon)$, then $g'' \geq 0$ on $[0, 1]$. Hence, g' is a non-decreasing function, so $g'(\xi) \leq g'(t)$. Therefore, $g(t) - g(0) \leq tg'(t)$. In particular, $g(1) - g(0) \leq g'(1)$; that is, $f(x) - f(u) \leq (x - u)^T \nabla f(x)$.

Since the invexity of f implies that $f(x) \geq f(u)$, then by the Cauchy–Schwarz inequality,

$$|f(x) - f(u)| \leq \left|(x - u)^T \nabla f(x)\right| \leq \|x - u\| \, \|\nabla f(x)\|.$$

Thus, if $\nabla f(x) \neq 0$, then $|f(x) - f(u)|/\|\nabla f(x)\| \leq \|x - u\|$.

Now, for any sequence $\{u^n\}$, $u^n \in C$, $u^n \to u$, $\nabla f(u^n) \neq 0$, there exists $M \in \mathbb{N}$ such that for all $n \geq M$ we have $u^n \in N(u, \epsilon)$, and consequently

$$\frac{|f(u) - f(u^n)|}{\|\nabla f(u^n)\|} \leq \|u^n - u\|.$$

Therefore, by the squeeze principle, $\lim_{n \to \infty} |f(u^n) - f(u)|/\|\nabla f(u^n)\| = 0$. ∎

4. NON-EXISTENCE OF A CONTINUOUS KERNEL

In general, in order to establish the nonexistence of a continuous kernel, there must exist $(x, u) \in C \times C$ and sequences $\{x^n\}$, $\{u^n\}$, $\{y^n\}$, $\{w^n\}$ with $x^n \to x$, $u^n \to u$, $y^n \to x$, $w^n \to u$ such that $\lim_{n \to \infty} \eta(x^n, u^n) \neq \lim_{n \to \infty} \eta(y^n, w^n)$ for any choice of $v(x^n, u^n)$ and $v(y^n, w^n)$. For the one-dimensional case, a condition can be obtained for functions with isolated minima which shows that in this situation Theorem 3.1 is both necessary and sufficient for the existence of a continuous kernel.

THEOREM 4.1. *Let C be an open interval in $\mathbb{R}$ and let $f: C \to \mathbb{R}$ be invex. If $\exists \bar{x} \in C$ such that $\bar{x}$ is an isolated minimum and*

$$\lim_{x \to \bar{x}} \frac{f(x) - f(\bar{x})}{f'(x)} \neq 0,$$

then there is no continuous η with respect to which f is invex.

Proof. Without loss of generality, we can assume that $\bar{x} = 0$ and $f(\bar{x}) = 0$. Now, whenever $f'(u) \neq 0$, any η must have the form

$$\eta(x, u) = \frac{f(x) - f(u)}{f'(u)} + v$$

where $vf'(u) \leq 0$.

Consider any $u \in C$ such that $u \neq 0$ and $f(u) > 0$ (and hence $f'(u) \neq 0$). We have

$$\eta(0, u) = -\frac{f(u)}{f'(u)} + v.$$

If $u > 0$, then the invexity of f implies $f'(u) > 0$, so $v \leq 0$, and so

$$\eta(0, u) \leq -\frac{f(u)}{f'(u)} < 0.$$

Similarly, if $u < 0$, then $f'(u) < 0$, so $v \geq 0$ and

$$\eta(0, u) \geq -\frac{f(u)}{f'(u)} > 0.$$

Now, by the limit assumption, there exists some sequence $\{u^n\}$, $u^n \to 0$, $f'(u^n) \neq 0$ such that $\lim_{n \to \infty} f(u^n)/f'(u^n)$ either (a) does not exist, or (b) exists and is non-zero. In either case, select any subsequence $\{u^{n_i}\}$ such that all terms are of the same sign. Without loss of generality, assume $u^{n_i} > 0$. Some category (a) sequences may have a category (b) subsequence, but we still have the same two categories to consider.

(a) As $-f(u^{n_i})/f'(u^{n_i}) < 0$ for all n_i and the limit does not exist, then $\lim_{n_i \to \infty} \eta(0, u^{n_i})$, if it exists, must be negative.

(b) $\lim_{n_i \to \infty} \eta(0, u^{n_i}) \leq \lim_{n_i \to \infty} -f(u^{n_i})/f'(u^{n_i}) < 0$, so the limit, if it exists, is negative.

However, $\lim_{u \uparrow 0} \eta(0, u) \geq 0$ if it exists, so $\lim_{u \to 0} \eta(0, u)$ cannot exist, and hence $\eta(0, \cdot)$ cannot be continuous at 0. ∎

It is possible to construct examples satisfying the criteria of Theorem 4.1 and which consequently do not have any continuous kernels.

EXAMPLE 4.1. Define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & x = 0 \\ \dfrac{n^2+n+1}{n+1}x^2 + \dfrac{-2n^2-2n-1}{(n+1)^2}x + \dfrac{4n^2+5n+2}{4(n+1)^3}, & \dfrac{1}{n+1} \le x \le \dfrac{2n+1}{2n(n+1)},\ n = 1,2,\dots \\ \dfrac{1-n^2}{n}x^2 + \dfrac{2n^2-1}{n^2}x + \dfrac{-4n^2+n+1}{4n^3}, & \dfrac{2n+1}{2n(n+1)} \le x \le \dfrac{1}{n}, \quad n = 1,2,\dots \\ x - \dfrac{1}{2}, & x \ge 1 \\ f(-x), & x < 0. \end{cases}$$

It is easy to check that f is continuously differentiable, with $f'(u) = 0$ iff $u = 0$, which is a global minimizer.

Consider the sequence $\{u^n\}$ with $u^n = 1/n$, $n = 1, 2, \dots$. We have

$$f(u^n) = \frac{n+1}{4n^3}$$

and

$$f'(u^n) = \frac{1}{n^2}.$$

Therefore,

$$\lim_{n\to\infty} \frac{|f(u) - f(u^n)|}{|f'(u^n)|} = \lim_{n\to\infty} \frac{n+1}{4n^3} \bigg/ \frac{1}{n^2} = \frac{1}{4}.$$

EXAMPLE 4.2. Consider $f\colon (-1,1) \to \mathbb{R}$ defined by

$$f(x) = \left(\sqrt{2} + (1 - x^3)\sin\left(\frac{1}{x^2}\right)\right)e^{-1/x^2}.$$

f is infinitely differentiable, and has an isolated global minimum at $x = 0$ with $f(0) = 0$. Observe that

$$f'(x) = \left(\sqrt{2} + \left(1 - x^3 - \frac{3x^5}{2}\right)\sin\left(\frac{1}{x^2}\right) - (1 - x^3)\cos\left(\frac{1}{x^2}\right)\right)\frac{2}{x^3}e^{-1/x^2}.$$

Define the sequence $\{u^n\}$ by $1/u^{n2} = 3\pi/4 + 2\pi n$, and note that $\sin(1/u^{n2}) = -1/\sqrt{2}$ and $\cos(1/u^{n2}) = 1/\sqrt{2}$.

Then,

$$f(u^n) = \left(\sqrt{2} - \frac{1}{\sqrt{2}}(1 - u^{n3})\right)e^{-1/u^{n2}}$$

$$f'(u^n) = \left(\sqrt{2} - \frac{2}{\sqrt{2}}(1 - u^{n3}) + \frac{3}{2}u^{n5}\right)\frac{2}{u^{n3}}e^{-1/u^{n2}}$$

so

$$\begin{aligned}
\frac{f(u^n)}{f'(u^n)} &= \frac{\sqrt{2} - (1/\sqrt{2})(1 - u^{n3})}{\left(\sqrt{2} - (2/\sqrt{2})(1 - u^{n3}) + (3/2)u^{n5}\right)(2/u^{n3})} \\
&= \frac{\sqrt{2} - (1/\sqrt{2})(1 - u^{n3})}{\left((2/\sqrt{2})u^{n3} + (3/2)u^{n5}\right)(2/u^{n3})} \\
&= \frac{\sqrt{2} - (1/\sqrt{2})(1 - u^{n3})}{(4/\sqrt{2}) + 3u^{n2}}.
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\frac{f(u^n)}{f'(u^n)} = \frac{\sqrt{2} - 1/\sqrt{2}}{4/\sqrt{2}} = \frac{1}{4}.$$

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